



Structured heterogeneity of network in evolutionary social dilemmas under assortative linking and learning

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ABSTRACT

We use assortativity in linking and learning by studying social dilemmas on a dynamic network, where assortativity in linking runs faster than assortativity in learning, so that both can work together. The emergence of cooperation is based on the structured heterogeneity of a network, which includes heterogeneous interactions and a heterogeneous population structure made by assortativity. In an infinite population, cooperation is more likely to happen in the Snowdrift game when there is heterogeneous interaction, in the Stag-hunt game when there is heterogeneous population structure, and in the Prisoner's Dilemma game when there are two different kinds of settings. In a finite population size, the heterogeneous interaction plays an important role in emergence of cooperation in all dilemmas.

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1. Introduction

Cooperation is one of the most intriguing issues now confronting scientists [1]. Cooperation is not favored by natural selection [2]. However, this prediction contradicts the many examples of cooperation in human societies. The homogeneous interaction setting is the primary cause of defection [3]. We introduce a new model where a heterogeneous population of agents are assortatively [4–6] matched in pairs to joint in a game. Each node (agents) carries one of two different strategy types. Network evolution is achieved by an assortative rewiring process. A random link between nodes A and B is chosen for assortative rewiring. Suppose node A is chosen for rewiring with a probability of a ($0 < a < 1$) of connecting to one of the nodes of the same strategy type in the network and a probability of $(1 - a)$ of randomly connecting to one of the nodes (i.e., preferential attachment). The connection between network evolution and preferential attachment dates back to Albert and Barabási [7], and networks are commonly referred to as scale-free networks. In this paper, a network with assortative preferential attachment is called an assortative network. This network model differs from those found in most previous studies [7–19] in which a node is more likely to meet a node linked with more neighbors.

To specify the assortative linking dynamics, we adhere to prior research [8]. To characterize the dynamics, a discrete stochastic model is considered. Assuming a network consists of two types of nodes (e.g., C and D); then, network rewiring due to assortativity results in a variable number of links (CC, CD, and DD). Therefore,

the system has three types of link existed. A certain type of link transforms into a link (CC, CD, or DD) with a given probability at each time step. In this case, the evolution of the network via assortative rewiring is a Markov chain. Given this, a transition probability matrix describing these links (CC, CD, and DD) is calculated; assuming the Markov chain is aperiodic and irreducible, a unique stationary distribution of dynamic links is then computed.

In this context, we also discuss assortative learning dynamics for the evolution of cooperation in the network. The method of assortative learning is similar to the process of assortative rewiring: at each time step, a random node is selected. A node has a b ($0 < b < 1$) probability of randomly adopting other nodes of the same kind and a $(1 - b)$ probability of adopting different types of nodes. When two nodes meet, they change their strategies using a *pairwise comparison rule* [9], namely one node adopting other node's strategy through strategy updating by comparing the payoffs between two nodes. Consequently, this paper presents two novel models: assortative linking dynamics and assortative learning dynamics.

Using the interaction between the two dynamics mentioned previously, we investigate the evolutionary process of games in a dynamic network where assortative linking dynamics proceed more quickly than assortative learning dynamics. This indicates that the dynamic network obtains dynamic stability first and then assortative learning dynamics are executed immediately (e.g., similar to the treatment in [10–13]). Here, we examine the role of assortative linking and learning in evolutionary social dilemmas in an effort to explain why cooperative behavior is so prevalent in human interactions. It is demonstrated that the structured heterogeneity of networks driven by assortativity

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in linking and initial number of cooperators separately favor the emergence of cooperation in different dilemmas in an infinite population, but are not necessary conditions for the emergence of cooperation in a finite population where heterogeneous interaction due to assortative linking can be of great significance.

2. Games and social dilemmas

Social dilemmas are situations in which individuals are willing to seek self-interest that leads to a lesser return than they would gain by promoting collective cooperation [14]. Social dilemmas are modeled as three types of two-person games in which players can either cooperate or defect in their encounters: the prisoner's dilemma game (PD), snowdrift game (SD), and stag-hunt game (SH). Games are classified as payoff structures with a combination of two actions, as seen in the table below. Mutual cooperation receives reward R , but mutual defection produces penalty P . Unilateral defection and cooperation lead to payoffs T and S , respectively. In terms of the relative ordering of payoffs R, P, T , and S , they naturally pose various dilemmas:

	C	D
C	R	S
D	T	P

In the PD scheme, for which $T > R > P > S, 2R > T + S$, unilateral defection is always optimal since the defector can earn reward T or P by ignoring their partner's choice—in this case, $T > R$ and $P > S$. Therefore, it is rational for both parties to defect. In the SD scheme, for which $T > R > S > P$, an individual's welfare is contingent on their partner's choices. Assuming the partner opts for defection, the person favors cooperation over defection since unilateral cooperation is superior than mutual defection for that individual, $S > P$. If the partner selects cooperation, they will pick defection since unilateral defection is superior than joint cooperation, $T > R$. Hence, individuals do not join into mutual cooperation. In the SH scheme, the payoff structure is $R > T > P > S$. Compared to the tension in the SD scheme, participants in the SH scheme share a similar reasoning, namely, that the welfare of the person depends on the choices of others, but with some distinctions. If the partner chooses defection, the individual will likewise defect, as unilateral cooperation is inferior than joint defection, $P > S$. In contrast, when the partner chooses cooperation, the person also chooses cooperation since joint cooperation is preferable than unilateral defection, $R > T$. Consequently, there is tension when individuals coordinate.

3. Role of assortative linking and learning in the evolution of cooperation

Consider the above three games played in a population of size N . Individuals are randomly put as nodes with L links in a network. This indicates that the initial number of links in the network is $LN/2$. The number of links possessed by an individual is tiny, suggesting a restricted number of network neighbors, while the overall number of links remains constant. The links are undirected and self-loops for an individual are banned.

We investigate how the frequencies of strategies C and D alter under the coevolution of linking and strategy in the three games. Consider the scenario in which the linking dynamics proceed sufficiently faster than strategy updating. As a result of the linking dynamics operating on a shorter period, the system establishes a steady state and then updates its strategy. Alternately, as the model of active linking proposes [15], if linking happens on a slower timeline strategy updating occurs on a static network, which is a common practice noted by previous studies [16].

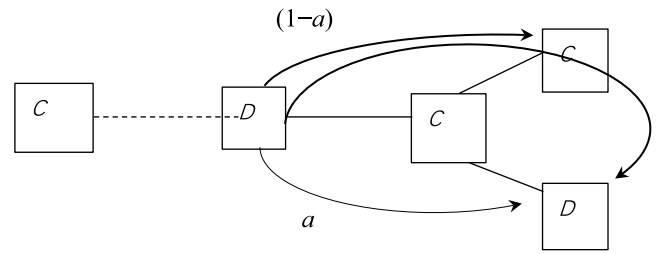


Fig. 1. Assortative linking. If dashed link CD is randomly chosen to be broken off, then, with a probability of $1/2$, end D is selected to perform the rewiring process. With a probability of a , end D connects to the same type held in the network; with a probability of $1 - a$, it randomly connects to an agent in the network.

The dynamics of links are represented as a rewiring process. In each phase of rewiring, a link ZK ($ZK = CC, CD, \text{ or } DD$) is randomly picked from the whole network to be severed, and then a random individual is chosen to rewire the link. A person with a probability of a ($0 < a < 1$) switches the link to one with the same strategy in the network, and then the link is rewired to an individual with a different strategy with a probability of $1 - a$ (see Fig. 1).

In this process of assortative linking, link ZK is eliminated and a new link ($CC, CD, \text{ or } DD$) is formed. Therefore, the dynamics of assortative linking turn out to be a Markov chain, and this process is described as a transition matrix, $M_{(ZK)(OP)}$, which is defined as the probability of link ZK transforming into a new link, OP . Intuitively, for links CC or DD , the probability of transforming from CC to DD or from DD to CC is zero. In other words, we can only calculate $M_{(CC)(CD)}, M_{(DD)(CD)}, M_{(CD)(CC)}$, and $M_{(CD)(DD)}$. As an illustration, $M_{(CC)(CD)}$, which is the probability that of link CC of t time transforms to link CD of $t + 1$ time, can be described as follows:

(1) Probability is $2/LN$ when link CC is randomly chosen in the initial step of the rewiring dynamics.

(2) When link CC is selected, the link is broken off and, with a probability of $1/2$, rewiring happens at one of the link's ends. Due to the presence of two ends of the same type, only when the selected end connects to a different type (i.e., with a probability of $1 - a$) will the link be transformed from CC to CD . Thus,

$$M_{(CC)(CD)} = \frac{2}{LN} \left[\frac{1}{2} (1 - a)(1 - x_c) \right] + \frac{2}{LN} \left[\frac{1}{2} (1 - a)(1 - x_c) \right] = \frac{2}{LN} (1 - a)(1 - x_c) \tag{1}$$

where x_c is the frequency of cooperators in the network.

Similarly, computing $M_{(DD)(CD)}, M_{(CD)(CC)}$, and $M_{(CD)(DD)}$ will give rise to the transition probability matrix:

$$T = \frac{LN - 2}{LN} I_3 + \frac{2}{LN} M$$

where I_3 is the identity matrix and matrix M is calculated as follows:

$$M = \begin{matrix} & \begin{matrix} CC & CD & DD \end{matrix} \\ \begin{matrix} CC \\ CD \\ DD \end{matrix} & \begin{pmatrix} 2/LN[a + (1 - a)x_c] & 2/LN(1 - a)(1 - x_c) & 0 \\ 1/LN[a + (1 - a)x_c] & 1/LN(1 - a) & 1/LN[a + (1 - a)(1 - x_c)] \\ 0 & 2/LN(1 - a)x_c & 2/LN[a + (1 - a)(1 - x_c)] \end{pmatrix} \end{matrix} \tag{2}$$

According to the ergodic theorem [20], when time approaches infinity, aperiodic and irreducible conditions will lead the Markov chain to attain a stationary distribution, as the transition probability is no longer equal to zero. Using the equation $\nu M = \nu$, we can compute a unique stationary distribution, $\nu = (\nu_{CC}, \nu_{CD}, \nu_{DD})$.

The result is

$$v = (x_c[a + (1 - a)x_c], 2x_c(1 - a)(1 - x_c), [a + (1 - a)(1 - x_c)](1 - x_c)) \tag{3}$$

Then, we examine the evolutionary process of the strategy as an illustration of assortative learning dynamics. It consists of two processes: (1) individual i is randomly chosen to adopt another with the same type with a probability of b ($0 < b < 1$) as well as to adopt a random agent (e.g., individual j) with a probability of $1 - b$; and (2) in the learning process, individual i adopts other individuals through strategy updating by comparing the payoffs between individual i and individual j . Individual i replaces their strategy with strategy j with the following probability [9]:

$$\frac{1}{1 + \exp^{\omega(\pi_i - \pi_j)}} \tag{4}$$

where π_i and π_j represent the accumulated payoff from the earlier assortative linking dynamics for i and j , respectively. Parameter ω represents the level of selection. For $\omega \leq 1$, selection is considered as a weakness; for $\omega > 1$, selection is a strength and it is more probable that an advantageous strategy will be taken.

4. Results and analysis

Strategy updating is significantly slower than that of the dynamics of assortative linking; hence, the average payoff accumulated from the interactions of each agent i hinges on the stationary state of the Markov chain in Eq. (3). Afterward, assortative learning dynamics are implemented. In this scenario, in each time step, there are three probabilistic cases for the number of cooperative individuals: adding a new cooperative member, maintaining the same, and losing an original one. The number of cooperators in the rise from N_c to $N_c + 1$ is decided by a *pair-wise comparison rule*, as given in Eq. (4), and the corresponding transition probability in such a situation is

$$T_c^+ = \underbrace{(1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}}}_A + \underbrace{(1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}}}_\chi = 2(1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}} \tag{5}$$

A is a situation in which a cooperator i is randomly chosen with probability $(1 - b)$ to meet a defector j randomly chosen, and the defector j takes the strategy of the cooperator i . χ is a situation in which a defector j is randomly chosen is with probability $(1 - b)$ to meet a cooperator j randomly chosen, and the defector j takes the strategy of the cooperator i .

Similarly, decreasing the total number of cooperators from N_c to $N_c - 1$ is described as

$$T_c^- = (1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} + (1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} = 2(1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} \tag{6}$$

We consider the heterogeneous network game under accumulated payoff framework. Using probability to increase the number of cooperators from N_c to $N_c + 1$, T_c^+ , and the probability to decrease the number from N_c to $N_c - 1$, T_c^- , the dynamics can be approximated by a stochastic differential equation with drift

$T_c^+ - T_c^-$ and diffusion $\sqrt{(T_c^+ + T_c^-)/N}$ [21] in a infinite population, and we can consider the effect of *fixation probability* to investigate how an individual with a different strategy can take over the entire population in a finite population.

For an infinite population size. In a large population, assortative learning dynamics may be approximated by a stochastic differential equation with drift $T_c^+ - T_c^-$ and diffusion $\sqrt{(T_c^+ + T_c^-)/N}$ [21]. Thus,

$$\begin{aligned} \dot{x}_c &= T_c^+ - T_c^- + \sqrt{(T_c^+ + T_c^-)/N} * \varepsilon \\ &= 2(1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}} - 2(1 - b)x_c x_D \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} \\ &\quad + \sqrt{(T_c^+ + T_c^-)/N} * \varepsilon \\ &= 2(1 - b)x_c(1 - x_c) \frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}} - 2(1 - b)x_c(1 - x_c) \\ &\quad \times \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} + \sqrt{(T_c^+ + T_c^-)/N} * \varepsilon \\ &= 2(1 - b)x_c(1 - x_c) \left(\frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}} - \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} \right) \\ &\quad + \sqrt{(T_c^+ + T_c^-)/N} * \varepsilon \end{aligned}$$

By using the calculations from the previous study [21], $\left(\frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_c^i)}} - \frac{1}{1 + \exp^{\omega(\pi_c^i - \pi_D^j)}} \right) = \tanh \frac{\omega}{2} (\pi_c^i - \pi_D^j)$, then this yields

$$\dot{x}_c = 2(1 - b)x_c(1 - x_c) \tanh \frac{\omega}{2} (\pi_c^i - \pi_D^j) + \sqrt{(T_c^+ + T_c^-)/N} * \varepsilon$$

where ε is the Gaussian white noise with variance 1.

In an infinite population, as N increases, the diffusion term $\sqrt{(T_c^+ + T_c^-)/N} * \varepsilon$ vanishes at $\sqrt{1/N}$, and the population satisfies a well-mixed condition. π_c^i and π_D^j can be approximated by π_c and π_D , which are, respectively, the average payoffs of the cooperators and defectors, respectively. Meanwhile, $\tanh \frac{\omega}{2} (\pi_c^i - \pi_D^j) = \frac{\omega}{2} (\pi_c^i - \pi_D^j) + O(x^3)$, where $O(x^3)$ is a higher order infinitesimal, yields

$$\dot{x}_c = \omega(1 - b)x_c(1 - x_c)(\pi_c - \pi_D). \tag{7}$$

Hence, we obtain a replicator equation to represent assortative learning dynamics to study the evolutionary dynamics of the strategy. Moreover, Eq. (7) is equivalent to the *payoff monotone dynamics* [2] method, which can be used to predict the equilibrium in a dynamic system by comparing the average payoffs of the cooperator and defector. Consequently, Eq. (7) consists of the difference between the expected payoff of the cooperator and that of the defector:

$$\eta(x_c) = \pi_c - \pi_D, \tag{8}$$

The π_c and π_D will be computed by the stationary distribution of the assortative linking dynamics:

$$\begin{aligned} \pi_c &= 2v_{CC}RLN/2Nx_c + v_{CD}SLN/2Nx_c = v_{CC}RL/x_c + v_{CD}SL/2x_c \\ \pi_D &= v_{CD}TLN/2Nx_D + 2v_{DD}PLN/2Nx_D = v_{CD}TL/2x_D + v_{DD}PL/x_D \end{aligned}$$

Further, from Eq. (3), and substituting v_{CC} , v_{CD} , and v_{DD} into the above equation, we obtain

$$\begin{aligned} \pi_c &= x_c[a + (1 - a)x_c]RL/x_c + 2x_c(1 - a)(1 - x_c)SL/2x_c \\ &= [a + (1 - a)x_c]RL + (1 - a)(1 - x_c)SL \end{aligned}$$

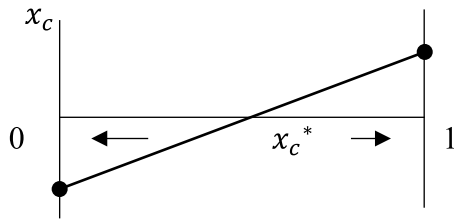


Fig. 2. Two stable equilibria in the evolutionary process.

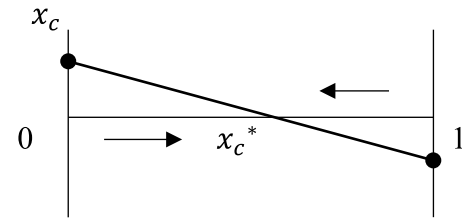


Fig. 3. Stable equilibria x_c^* in the evolutionary process.

$$\begin{aligned} \pi_D &= 2x_c(1-a)(1-x_c)TLN/2Nx_D + [a+(1-a)(1-x_c)] \\ &\times (1-x_c)PL/x_D \\ &= x_c(1-a)TL + [a+(1-a)(1-x_c)]PL \end{aligned}$$

Substituting the equations above into Eq. (8), we have

$$\eta(x_c) = a(R-S) + S - P + (1-a)(R+P-S-T)x_c \quad (9)$$

Therefore, the payoff monotone dynamics are determined by the signs of

$$\eta(0) = a(R-S) + S - P \quad (10)$$

$$\eta(1) = R - T + aT - aP \quad (11)$$

Proposition 1. In the PD ($T > R > P > S$, $2R > T + S$):

(1) There is a unique stable equilibrium $x_c = 0$ if and only if $0 < a < \frac{(T-R)}{(T-P)}$.

(2) There is a unique stable equilibrium $x_c = 1$ if and only if $\frac{(T-R)}{(T-P)} < a < 1$ and $x_c > \frac{P-S-a(R-S)}{(1-a)(R+P-S-T)}$.

Because $\eta(x_c)$ is linear in x_c , when $\eta(0) < 0$ and $\eta(1) < 0$ (i.e., when $(R-S) + S - P < 0$ and $R - T + aT - aP < 0$), $\eta(x_c)$ is always less than zero for any x_c , such that cooperators always have a smaller expected payoff than defectors, and there is a unique stable equilibrium x_c . In this case, according to $a(R-S) + S - P < 0$ and $R - T + aT - aP < 0$, we have $a < \frac{(T-R)}{(T-P)}$ and $a < \frac{(R-S)}{(P-S)}$. In the PD scheme, the payoff structure $T > R > P > S$ and $2R > T + S$ yield $0 < \frac{(T-R)}{(T-P)} < 1 < \frac{(R-S)}{(P-S)}$. Because $0 < a < 1$, we have $0 < a < \frac{(T-R)}{(T-P)}$, and this yields Proposition 1(1).

Similarly, when $\eta(0) < 0$ and $\eta(1) > 0$, as shown in Fig. 2, the expected payoff of cooperators is larger than that of defectors ($x_c > x_c^*$) and the expected payoff of cooperators is smaller than that of defectors as $x_c < x_c^*$. This means that when cooperators are rare, $x_c = 1$ in a stable equilibrium; otherwise, when defectors are rare, $x_c = 0$ in a stable equilibrium. Hence, there are two stable equilibria: $x_c = 0$ and $x_c = 1$. In this case, given $\eta(0) < 0$ and $\eta(1) > 0$, we have $a < \frac{(R-S)}{(P-S)}$ and $a > \frac{(T-R)}{(T-P)}$, so that $\frac{(T-R)}{(T-P)} < a < 1$ and $x_c > x_c^*$ are found. x_c^* means $\eta(x_c) = 0$, so that $x_c^* = \frac{P-S-a(R-S)}{(1-a)(R+P-S-T)}$, thus giving Proposition 1(2).

Proposition 2. In the SD ($T > R > S > P$):

(1) There is a unique stable equilibrium $x_c = 1$ if and only if $\frac{(T-R)}{(T-P)} < a < 1$;

(2) There is a unique stable equilibrium $x_c = \frac{P-S-a(R-S)}{(1-a)(R+P-S-T)}$ if and only if $0 < a < \frac{(T-R)}{(T-P)}$.

When $\eta(0) > 0$ and $\eta(1) > 0$, the expected payoff of a cooperator is greater than that of a defector in all evolutionary processes, which means that there is a unique stable equilibrium, $x_c = 1$. In this case, with the conditions $\eta(0) > 0$ and $\eta(1) > 0$, we have $a > \frac{(T-R)}{(T-P)}$ and $a > \frac{(R-S)}{(P-S)}$. In the SD, we have $T > R > S > P$, and $1 > \frac{(T-R)}{(T-P)} > 0 > \frac{(R-S)}{(P-S)}$ is

obtained; thus, when $\frac{(T-R)}{(T-P)} < a < 1$, there is a unique stable equilibrium that contains only cooperators.

When $\eta(0) > 0$ and $\eta(1) < 0$ (see Fig. 3), there is a unique stable equilibrium $x_c = x_c^*$ ($0 < x_c^* < 1$), which is determined by the evolutionary process. When cooperators are rare, the expected payoff of a cooperator is higher than that of a defector and the number of cooperators increases. When defectors are rare, the expected payoff of cooperators is smaller than that of defectors and the number of cooperators decreases. Thus, there is a unique stable equilibrium, $x_c = x_c^*$ ($0 < x_c^* < 1$), which includes some cooperators and defectors. In this case, when $\eta(0) > 0$ and $\eta(1) < 0$, we have $0 < a < \frac{(T-R)}{(T-P)}$, and when $\eta(x_c) = 0$, we have $x_c^* = \frac{P-S-a(R-S)}{(1-a)(R+P-S-T)}$, thus giving Proposition 1(2).

Proposition 3. In the SH ($R > T > P > S$):

There is unique stable equilibrium $x_c = 1$ if and only if $0 < a < 1$ and $x_c > \frac{P-S-a(R-S)}{(1-a)(R+P-S-T)}$.

Proof. The logic behind the proof of Proposition 3 is the same as that for Proposition 1(2).

Together with the propositions above, we can conclude the following:

The results shown in Fig. 4 explain the coevolution of assortative linking dynamics and assortative learning dynamics in an infinite population. These findings reveal the role of assortative rewiring and show the conditions under which cooperation might emerge. The increase in the degree of assortative rewiring (i.e., a greater likelihood of rewiring to an agent of the same type at each time step) facilitates the emergence of cooperation in the SD. Specifically, for the PD and SG schemes, the equilibrium—the coexistence of cooperators and defectors—is so unstable that divergence develops as the number of cooperators changes. When the number of cooperators is common in the population, cooperation dominates defection in the PD, as the degree of assortative rewiring increases, as well as in the SG, where the role of assortative rewiring is overlooked. Consequently, the results reveal three cases among social dilemmas in which the emergence of cooperation depends only on the degree of assortative rewiring (in the SD) or on the number of cooperators initially (in the SG), and on the combination of both (in the PD).

Eq. (7) shows that b has no effect the evolutionary equilibria in the replicator dynamics, whereas it does impact the speed of movement toward the equilibria. As this speed slows (i.e., as individuals with the same strategy occur less frequently than those with different strategies), together with Proposition 2(1), the frequency of cooperation in the SG scheme gets closer to one, accelerating the convergence of the equilibria. In the meantime, it also accelerates the equilibrium to be a defective equilibrium (see Proposition 1(1)). The parameter b is hence neutral. The role of w in the replicator dynamics is the same as the function of b , except it operates in the opposite direction (i.e., it accelerates convergence as it changes). A strong w exerts pressure to accelerate the convergence of the equilibria, whereas a weak w retards

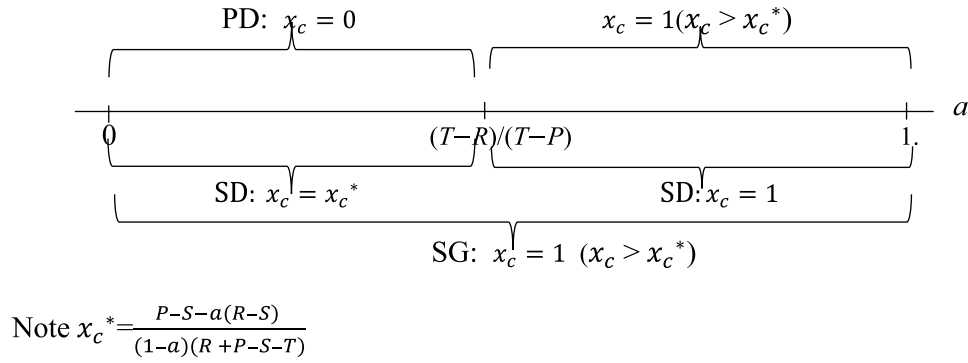


Fig. 4. Evolutionary equilibria as a function of a .

the convergence of the equilibria. In this context, an advantaged strategy is more likely to dominate the whole population through a strong selection.

For a finite population size. In a finite large population, we address fixation probability following previous studies [3,20,22–25]. We attempt to ascertain the possibility that a person employing with a certain strategy will invade and take over the population. Let ρ_C represent the probability that a cooperater has entered a population of defectors in order to take over the entire population and ρ_D be a defector who aims to take over the entire population of cooperators. In previous studies [3,22–25], ρ_C and ρ_D were characterized as follows:

$$\rho_C = \frac{1}{1 + \sum_j^{N-1} \prod_i^j \gamma_i}, \quad \rho_D = \frac{\prod_i^{N-1} \gamma_i}{1 + \sum_j^{N-1} \prod_i^j \gamma_i} \quad (12)$$

where

$$\gamma_i = \frac{T_C^-}{T_C^+}.$$

According to Eq. (12), we have

$$\begin{aligned} \frac{\rho_D}{\rho_C} &= \left(\frac{\prod_i^{N-1} \gamma_i}{1 + \sum_j^{N-1} \prod_i^j \gamma_i} \right) / \left(\frac{1}{1 + \sum_j^{N-1} \prod_i^j \gamma_i} \right) \\ &= \prod_i^{N-1} \gamma_i \\ &= \prod_i^{N-1} \frac{T_C^-}{T_C^+}. \end{aligned} \quad (13)$$

If $\rho_D/\rho_C > 1$, defectors take advantage of cooperators in the evolutionary process. Using the calculations in Eqs. (5) and (6) yields

$$\begin{aligned} \frac{T_C^-}{T_C^+} &= 2(1-b)x_C x_D \left(\frac{1}{1 + \exp^{\omega(\pi_C^i - \pi_D^i)}} \right) / (2(1-b)x_C x_D \\ &\quad \times \frac{1}{1 + \exp^{\omega(\pi_D^j - \pi_C^j)}}) \\ &= \frac{1 + \exp^{\omega(\pi_D^j - \pi_C^j)}}{1 + \exp^{\omega(\pi_C^i - \pi_D^i)}} \approx \exp^{-\omega(\pi_C^i - \pi_D^j)} \end{aligned} \quad (14)$$

In a large population in which individuals are well mixed, $\pi_C^i - \pi_D^j$ is equal to $\pi_C - \pi_D$. Substituting Eq. (9) into Eq. (14) yields

$$\frac{\rho_D}{\rho_C} = \prod_i^{N-1} \frac{T_C^-}{T_C^+}$$

$$\begin{aligned} &= \exp^{-\omega \sum_i^{N-1} (\pi_C - \pi_D)} \\ &= \exp^{-\omega \sum_i^{N-1} (a(R-S) + S - P + (1-a)(R+P-S-T) \frac{i}{N})} \\ &= \exp^{-\omega [a(R-S)(N-1) + (S-P)(N-1) + (1-a)(R+P-S-T) \frac{1}{N} \frac{N(N-1)}{2}]} \end{aligned}$$

For $\mathcal{E} > 0$, we have $\rho_C > \rho_D$. Therefore,

$$\begin{aligned} \mathcal{E} &> 0 \\ &\rightarrow a(R-S)(N-1) + (S-P)(N-1) + (1-a)(R+P-S-T) \\ &\quad \times \frac{1}{N} \frac{N(N-1)}{2} > 0 \\ &\rightarrow a > \frac{T-R+P-S}{R-S-P+T} = \frac{(T-S) - (R-P)}{(T-S) + (R-P)} = \underbrace{\frac{1 - \frac{(R-P)}{(T-S)}}{1 + \frac{(R-P)}{(T-S)}}}_{\Phi}. \end{aligned} \quad (15)$$

Proposition 4. (1) In the PD, cooperators dominate defectors if and only if $\Phi < a < 1$;

(2) In the SD and SH, cooperators dominate defectors if and only if $0 < \frac{(R-P)}{(T-S)} < 1$ and $\Phi < a < 1$.

In the PD, we have $T > R > P > S$ and $2R > T + S$ such that $0 < (R-P)/(T-S) < 1$. Because $0 < a < 1$, when $\Phi < a < 1$, Eq. (15) is supported, and a cooperater invades the population of defectors and takes over this entire population, namely, $\rho_C > \rho_D$. Thus, Proposition 4(1) can be proven.

In the SD, we have $T > R > S > P$. Compared with the PD, it is uncertain whether $0 < (R-P)/(T-S) < 1$, and only if $0 < (R-P)/(T-S) < 1$ do we have $0 < \Phi < 1$, and then $\Phi < a < 1$ can be met. When $0 < (R-P)/(T-S) < 1$ and $\Phi < a < 1$, cooperators are supported in the evolutionary process of the SD. Similar considerations apply to the SH, and Proposition 4(2) holds.

In Proposition 4, we can also focus on the role of ω . For $\omega = 0$, Eq. (14) will be equal to 1, this means ρ_C will be equivalent to ρ_D , Proposition 4 will be not valid. For $\omega = 1$, it corresponds to strong selection and yields an accelerating speed to make a cooperater to take over the entire population in Proposition 4. For $0 < \omega < 1$, it corresponds to weak selection and yields slowing speed of results in Proposition 4.

5. Discussion and conclusion

In this study, the coevolution of assortativity in linking and learning is investigated. We provide a novel model for networks based on assortative linking, in which people are placed in various social dilemmas. To represent assortative linking for the

evolution of networks, we present a discrete stochastic model in which the frequencies of various types of links are governed by a Markov chain, and these ultimately form networks with structured heterogeneity. The evolution of cooperation in a stochastic assortative network is then achieved. This finding extends the use of heterogeneous graph theory [10,20,23,24] to model evolutionary cooperation. In previous studies, more dilemmas of heterogeneous networks have been considered, such as a small-world network [25] and a scale-free network [3]. Therefore, relevant games require further study in stochastic assortative networks.

To examine the evolution of cooperation, we also employ a two-layer network approach [26–29], consisting of an interaction network and a learning network. Ohtsuki et al. [26] initially presented this concept to examine the evolution of cooperation, claiming that it happens in a two-layer network. One layer is an interaction network in which links are formed by the games players play, and the other layer is a learning network built by learning dynamics. Our study contributes to both the interaction network and the learning network models. In particular, the bulk of prior studies have solely examined imitation dynamics through random learning, ignoring assortative learning. Because we know that people are willing to adopt a person of the same type and expect a higher payoff, assortative learning dynamics shed light on this phenomenon.

Numerous experts have studied and researched the segregation phenomena [30–33]. We analyze a portion of this segregation in terms of the evolution of cooperation in the network, but not the hyper ($a = 1$) aspect. Although hyper-segregation is uncommon, it may occur in some religious communities. It is reasonable for an individual to match with a person who conducts the same actions. However, bounded rationality, higher search costs, and the presence of individuals with incomplete information on their type all prevent complete segregation. Our study concentrates on a typical reality.

In this study, heterogeneous interaction between network nodes was achieved through an assortative rewiring process. This process is a preferential attachment mechanism. Previous studies have reported that preferential attachment can trigger assortativity, which affects evolutionary dynamics markedly [15,34,35]. For example, using the social leverage interaction, Zhang and Elsner [34] created a social leverage network in which, through a common acquaintance, the probability that cooperative agents encounter cooperators is increased, thus rendering interactions non-random (i.e., the *assortative matching* process). The results of their evolutionary dynamics are similar to the findings of this study. Pacheco, Traulsen, and Nowak [35] focused on the role of active linking in evolutionary games, where individuals differ in the rate at which they take new interactions, and links can be broken off at different rates, so that in some senses in a certain setting cooperative agents encounter cooperators is increased. The common feature of the two studies explains how interaction becomes assortative, generating a stable evolution of cooperation. However, the advantage of the present study is that it considered an assortative rewiring process instead of a social leverage process and used a two-layer network approach to explore the role of assortative linking and learning in evolutionary social dilemmas. In addition, the different social dilemmas presented different outcomes of evolutionary cooperation, which is interesting and enlightening.

We neglected cultural assortativity and other types of assortativity, which remains an opportunity for future studies. If we consider cultural assortativity in rewiring linking, a heterogeneous population would be considered; in other words, the evolution of cooperation would occur in two populations with

different cultural identities. Such a study could thus explain the evolution of cooperation among cultural groups.

Declaration of competing interest

There is no conflicts to declare.

Data availability

No data was used for the research described in the article.

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